SEMANTICAL TRANSLATIONS FOR MODAL LOGICS BASED ON BOOLEAN FUNCTION THEORIES

MIGUEL COUCEIRO
DEPARTMENT OF MATHEMATICS, STATISTICS AND PHILOSOPHY
UNIVERSITY OF TAMPERE, FINLAND

The language of modal logic is obtained from the language of propositional logic by adding a modal operator $\Box$: formulas are build up from propositional symbols $p \in \Phi$, using propositional connectives $\neg$, $\wedge$, and the unary operator $\Box$.

A *Kripke frame* is a structure $\mathbf{R} = \langle W, R\rangle$ where W is a non-empty set and $R \subseteq W^2$. These frames provide neat semantics for the so-called *normal systems* of modal logic. A *Kripke model* is a structure $\mathbf{M} = \langle \mathbf{R}, V\rangle$ where $\mathbf{R}$ is a Kripke frame and V is a function $\Phi \to \mathcal{P}(W)$. The truth conditions are defined recursively as in propositional logic, except for the modal operator $\Box$:

$$\mathcal{M}, w \models \Box\varphi \Leftrightarrow \forall v \in W, wRv \Rightarrow \mathcal{M}, v \models \varphi$$

But Kripke models cannot account for non-normal modal logics (e.g. *classical systems*). The Kripke semantics can be generalized by means of *Scott-Montague frames*, i.e. structures $\mathbf{F} = \langle W, F\rangle$ where F is a set function $\mathcal{P}(W) \to \mathcal{P}(W)$. The truth condition for the modal operator $\Box$ in a Scott-Montague model $\mathbf{M} = \langle \mathbf{F}, V\rangle$ is given by: $\mathbf{M}, w \models \Box\varphi \Leftrightarrow w \in F(||\varphi||)$, where $||\varphi|| = \{v \in W \,|\, \mathbf{M}, v \models \varphi\}$. For further background see e.g. [**?**].

We say that a class $\mathcal{K}$ of Kripke frames *corresponds* to a class $\mathcal{F}$ of Scott-Montague frames, if there are translations $R \mapsto F_R$ and $F \mapsto R_F$ such that

- if $\langle W, R\rangle \in \mathcal{K}$, then $\langle W, F_R\rangle \in \mathcal{F}$, and if $\langle W, F\rangle \in \mathcal{F}$, then $\langle W, R_F\rangle \in \mathcal{K}$,
- for all modal formulas φ, $\langle W, R\rangle \models \varphi \Leftrightarrow \langle W, F_R\rangle \models \varphi$, and $\langle W, F\rangle \models \varphi \Leftrightarrow \langle W, R_F\rangle \models \varphi$,
- $R_{F_R} = R$ and $F_{R_F} = F$.

Let $\mathbb{B} = \{0, 1\}$. Several important classes $\mathcal{C} \subseteq \bigcup_{n\geq 1} \mathbb{B}^{\mathbb{B}^n}$ of Boolean functions are known to be definable by *functional terms*, i.e. formal expressions $h(\mathbf{f}(g_1(\mathbf{v}_1, \ldots, \mathbf{v}_p)), \ldots, \mathbf{f}(g_m(\mathbf{v}_1, \ldots, \mathbf{v}_p)))$, where $h : \mathbb{B}^m \to \mathbb{B}$, each g_i is a map $\mathbb{B}^{pn} \to \mathbb{B}^n$, the $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vector variable symbols, and $\mathbf{f}$ is a function symbol. In particular, Boolean clones, i.e. classes containing all projections and closed under composition, are definable by functional terms (see [**?**]).

In this talk we show that several natural classes of Kripke frames correspond to classes Scott-Montague frames which are "determined" by Boolean clones. By means of suitable translations between functional terms and modal formulas of a prescribed syntax (*uniform* 1*-degree*), and using the natural bijection between set functions $F : \mathcal{P}(W) \to \mathcal{P}(W)$ and Boolean operators $f : \mathbb{B}^n \to \mathbb{B}^n$, we establish a complete correspondence between classes definable by functional terms and classes of Scott-Montague frames axiomatizable by uniform 1-degree formulas. This correspondence is used to determine the desired translations $R \mapsto F_R$ and $F \mapsto R_F$.

The results discussed in this presentation were obtained jointly with L. Hella and J. Kivelä at University of Tampere.

References

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E-mail address: Miguel.Couceiro@uta.fi